GCE: Analysis, measure theory, Lebesgue integration

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Exercise 1:

Let f_n be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$ which is uniformly convergent.
Let x_n be in $[0,1]$ such that $f_n(x_n) \geq f_n(x)$, for all x in $[0,1]$.
(i). Is the sequence x_n convergent?
(ii). Show that the sequence $f_n(x_n)$ is convergent.

Exercise 2:

Let $\mathbb{I}$ be the set of all irrational real numbers ($\mathbb{I} \subset \mathbb{R}$).

1. Using that $\mathbb{Q} = \mathbb{R} \setminus \mathbb{I}$ (the set of all rationals) is countable, show that given $\epsilon > 0$, there is a closed subset $F \subset \mathbb{I}$ such that $|\mathbb{I} \setminus F| < \epsilon$.
   Here $|A|$ is the Lebesgue measure of the set A, and $\setminus$ is set subtraction.

2. Is F compact? Please explain why or why not.

Exercise 3:

Find (with proof),

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^3}{(1+x^2)^n} dx$$

Exercise 4:

Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let f be in $L^1(X)$ such that $f \geq 0$ almost everywhere.
(i). Show that

$$\lim_{p\to 0^+} \int f^p = \mu(\{x \in X : f(x) > 0\})$$

(ii). If $\mu(\{x \in X : f(x) > 0\}) < 1$, find

$$\lim_{p\to 0^+} (\int f^p)^{1/p}.$$